

Error Analysis and Comparison of Two Algorithms Measuring Compensated Income

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Abstract We analyze and compare the errors of two numerical approaches for measuring compensated income. We prove that Vartia's algorithm and Breslaw and Smith's algorithm both converge quadratically; when the price change within each partition step is small, the error of Vartia's algorithm is approximately half that of Breslaw and Smith's algorithm. A theorem and the appropriate simulations with three different demand systems correct the existing error analysis of the two algorithms, and demonstrate that Vartia's algorithm is more accurate than Breslaw and Smith's.

Keywords Numerical method · Convergence rate · Compensating variation · Vartia's algorithm · Breslaw and Smith's algorithm

1 Introduction

Compensating variation (CV) and equivalent variation (EV) are frequently used as measures of the effect of a change in state on a consumer's welfare. Among the numerical approaches used to estimate these measures, two algorithms are widely discussed and applied: the Main Algorithm of [Vartia \(1983\)](#) (hereafter Vartia) and an alternative proposed by [Breslaw and Smith \(1995\)](#) (hereafter B&S). These two algorithms divide a price change into numerous small partition steps. The approximated measures estimated with the algorithms converge to the true measures as the number of partition steps increases.

An important feature of a numerical method is the convergence rate, as it determines the accuracy and efficiency of the algorithm. On the general importance of proving the

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convergence rate of algorithms, Santos and Vigo-Aguiar (1998) point out that “... it is important to have in hand error bounds or accuracy estimates of the computed solutions. As is commonly realized (e.g., Judd 1992) there has been very little theoretical work by economists on proving the accuracy of their numerical simulations.” To our knowledge, no rigorous and complete proof of the convergence rate exists for Vartia’s and B&S’s algorithms. Vartia claims in his Theorem 1 that his algorithm converges cubically, but his proof has mistakes. The cubic convergence rate, however, is widely accepted in literatures (e.g., Hayes and Porter-Hudak 1987b; Balk 1995; Hausman and Newey 1995). B&S simulate Vartia’s algorithm and assert that it converges linearly, but the code that they use contains errors. B&S also present their own algorithm and show, by simulation, that it converges quadratically. They conclude that their algorithm is more accurate and efficient than Vartia’s.

In this note, we present a graphical comparison of the two algorithms using a one-step example in which Vartia’s algorithm is deemed more accurate. The example motivates a more rigorous mathematical treatment. We prove a theorem that both algorithms converge quadratically, and that the error of Vartia’s algorithm is approximately half that of B&S’s algorithm, when the price change within each partition step is small. Simulations on one-good and multi-good demand systems verify the theorem and support the conclusion that Vartia’s algorithm is more accurate.

2 Algorithm Background and Illustrative Analysis

We follow common definitions in demand theory. The function $u(x)$ is the utility function, where x is the consumption bundle. The function $q(p, C)$ is the Marshallian demand function, where p is price and C is income. The Marshallian demand function is considered well-behaved; that is, for a complete demand system, the usual integrability condition is satisfied, and for an incomplete demand system, the weak-integrability condition (see LaFrance and Hanemann 1989) is satisfied. Furthermore, the function $q(p, C)$ is thrice continuously differentiable. The function $v(p, C)$ is the indirect utility function, and the function $h(p, \bar{v})$ is the Hicksian demand function, where \bar{v} is the required utility level.

Suppose the price changes from p^0 to p^1 . The compensated income C^1 for the consumer to maintain his initial utility level is defined implicitly by the equation

$$v(p^1, C^1) = v(p^0, C^0). \quad (1)$$

The CV is the difference between C^0 and C^1 , and we calculate it by the equation

$$CV = - \int_{p^0}^{p^1} h(p, u^0) dp = - \int_{p^0}^{p^1} q(p, C(p)) dp, \quad (2)$$

where $C(p)$ is the compensated income holding the consumer’s utility at the initial level u^0 .

To calculate the CV, we need the Hicksian demand function. In practice, we can only retrieve information available from the Marshallian demand function. Hausman (1981) derives the exact CV and EV for some Marshallian demand specifications with a single price change. However, when the Marshallian demand functions are complex and two or more prices change, Hausman (1981)'s method is difficult to implement because it requires solving differential equations. This difficulty encourages researchers to approximate the integral of the Hicksian demand with information derivable from the Marshallian demand. The methods that have been developed fall into two main categories: one-step and numerical.

An early example of a one-step method is that developed by Malmquist (1993) in the 1950s: he estimates the compensated income with a trapezoidal approximation.¹ McKenzie and Pearce (1976) and Mas-Colell et al (1995) use a different idea from Malmquist's to estimate the CV: they expand the integral of the Hicksian demand at the initial point to the second order with the Slutsky equation. Another attractive method is proposed by Irvine and Sims (1998), who use the Slutsky compensated demand through the initial point to approximate the Hicksian demand and estimate the CV.

Although these one-step approximations are easy to implement, their errors cannot be controlled. When the price change is large, the CV calculated from these algorithms will be inaccurate. A related practical problem arises when the true CV or EV is close to zero, in which case these one-step estimators may wrongly suggest the sign of welfare change. Numerical approaches solve the accuracy problem with rather simple algorithms: first, divide the price change into n steps; second, approximate the CV (i.e., the integral of the Hicksian demand) within each small step; finally, aggregate over all the steps to obtain the total CV for the whole price change. In the following two subsections we introduce the numerical approaches proposed by Vartia and B&S, which are easy to implement and thus widely used.

2.1 Vartia's Algorithm

Suppose the initial income is C^0 and the price changes from p^0 to p^1 . Divide the price change into n steps. Let $p_k = p^0 + \frac{k}{n}(p^1 - p^0)$ be the ending price at step k , where $k = 1, \dots, n$. Within each partition step from p_{k-1} to p_k , Vartia's algorithm first finds a converging ending point $q(p_k, C_k)$ at the ending price p_k ; second, it approximates the integral of the Hicksian demand as the trapezoidal area between the two prices formed by the initial point and the ending point.² Therefore, the algorithm generates a sequence C_1, \dots, C_n so that

$$C_k - C_{k-1} = \frac{1}{2}(q_k + q_{k-1}) \cdot \Delta p_k, \quad (3)$$

¹ Malmquist (1993) was written in the 1950s but published in 1993. The idea is reconstructed in Balk (1995).

² As Balk (1995) points out, the idea of Vartia's approximation is similar to that in Malmquist (1993).

where $q_k = q(p_k, C_k)$, $q_{k-1} = q(p_{k-1}, C_{k-1})$, $\Delta p_k = p_k - p_{k-1}$, and the starting values are $(p_0, q_0, C_0) = (p^0, q(p^0, C^0), C^0)$. The solution C_k is determined by iteration from

$$C_k^m = C_{k-1} + \frac{1}{2} \left(q(p_k, C_k^{m-1}) + q_{k-1} \right) \cdot \Delta p_k, \quad (4)$$

where $C_k^0 = C_{k-1}$. When $|C_k^m - C_k^{m-1}|$ is negligible, set $C_k = C_k^m$ and calculate for the next k . The estimator C_n converges to the compensated income C^1 as the number of partition steps n increases.

The algorithm is widely cited and well developed. The variance of the estimator has been calculated by [Porter-Hudak and Hayes \(1986, 1991\)](#) and [Hayes and Porter-Hudak \(1987a\)](#). An application and extension to the oil market can be found in [Hayes and Porter-Hudak \(1987b\)](#).

2.2 Breslaw and Smith's Algorithm

B&S use an idea similar to that in [McKenzie and Pearce \(1976\)](#) and [Mas-Colell et al \(1995\)](#) and offer their own algorithm approximating the integral of the Hicksian demand with second-order Taylor expansion and the Slutsky equation. Suppose the initial income is C^0 and the price changes from p^0 to p^1 . Divide the price change into n steps. Let $p_k = p^0 + \frac{k}{n}(p^1 - p^0)$ be the ending price at step k , where $k = 1, \dots, n$. The algorithm generates a sequence C_1, \dots, C_n so that

$$C_k - C_{k-1} = q_{k-1} \Delta p_k + \frac{1}{2} \left(\frac{\partial q_{k-1}}{\partial p} + \frac{\partial q_{k-1}}{\partial C} q_{k-1} \right) (\Delta p_k)^2, \quad (5)$$

where $q_{k-1} = q(p_{k-1}, C_{k-1})$, $\Delta p_k = p_k - p_{k-1}$, and the starting values are $(p_0, q_0, C_0) = (p^0, q(p^0, C^0), C^0)$. The estimator C_n converges to the compensated income C^1 as the number of partition steps n increases.

Their algorithm draws much interest and discussion. [Irvine and Sims \(2002\)](#) discuss applications of the algorithm, a recent application to the gasoline market can be found in [Shin and Burke \(2010\)](#). [Dumagan and Mount \(1997\)](#) extend the algorithm by first expanding the integral of the Hicksian demand at the initial point and the ending point and then averaging the two expansions to get the approximated CV.

2.3 Relative Size of the Errors: A One-Step Example

We compare the errors and convergence rates of Vartia's and B&S's algorithms with respect to n , the number of partition steps. Theoretically, given n , the algorithm with smaller errors is more accurate. Indeed, the smaller error and higher rate of convergence with respect to n are the main reasons B&S claim their algorithm dominates Vartia's in convergence performance.

To illustrate the two algorithms and our idea, we consider an example in which $u = x_1 x_2$, $C^0 = 1$, p_1 changes from 3.0 to 1.0, and p_2 is fixed at 1.0. We calculate the compensated income C^1 by the two algorithms with only one partition step ($n = 1$).

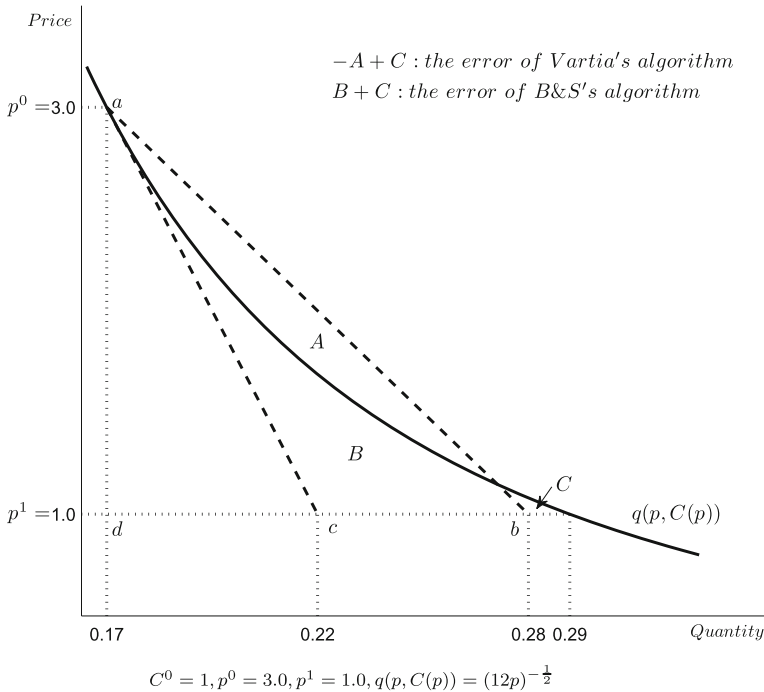


Fig. 1 Errors of the two algorithms: a one-step example

Figure 1 is the quantity-price span of the first good.³ Given a price change from $p^0 = 3.0$ to $p^1 = 1.0$, the curve $q(p, C(p))$ is the Hicksian demand holding the utility at the initial level. According to Eq. (2), the true CV is the area on the left of the curve $q(p, C(p))$ between $p^0 = 3.0$ and $p^1 = 1.0$. We write $\Delta p = p^1 - p^0 = -2.0$.

After the iteration, the CV calculated by Vartia’s algorithm is the area on the left of the line ab between $p^0 = 3.0$ and $p^1 = 1.0$, where the point $b = (p^1, q(p^1, C_1^V))$ satisfies

$$C_1^V = C^0 + \frac{1}{2} \left(q \left(p^1, C_1^V \right) + q \left(p^0, C^0 \right) \right) \Delta p, \tag{6}$$

which is the converging version of Eq. (4). Therefore, the error of Vartia’s algorithm is the area $-A + C$.

The CV calculated by B&S’s algorithm is the area on the left of the line ac between $p^0 = 3.0$ and $p^1 = 1.0$. Line ac is tangent to $q(p, C(p))$ at the initial point $(p^0, q(p^0, C^0))$. This can be shown by Eq. (5): the first term is the rectangular area on the left of ad between $p^0 = 3.0$ and $p^1 = 1.0$. Since $(p_0, q_0) = (p^0, q(p^0, C^0))$, the second term equals $\frac{1}{2} \frac{dq(p^0, C^0)}{dp} (\Delta p)^2$, which is the area of the triangle formed by a, c , and d . Therefore, the error of B&S’s algorithm is the area $B + C$.

³ All figures are developed with MATLAB, and are presented to scale.

Figure 1 provides three important observations. First, the point b is close to the true compensated point $(p^1, q(p^1, C(p^1)))$, so the area C is small. Therefore, the relative size of the errors depends largely on the relative size of the areas A and B . Second, $B > A$ and thus $|B + C| > |-A + C|$, which means Vartia's algorithm has a smaller error. Third, the errors have opposite signs.

The above observations put in doubt B&S's claim that their algorithm is more accurate. Indeed, the example motivates us to look deeper into the relationship between the errors of the two algorithms in more general cases.

3 A Theorem on the Convergence Rates and Relative Size of the Errors

Vartia's Theorem 1 states that the total error over n steps converges cubically with respect to n , the total number of partition steps; his Appendix A, however, only claims that the error of his algorithm over k steps converges cubically with respect to n . As we will show later, the total error over n steps and error over k steps are different, and they converge at different rates with respect to n .

B&S present numerically that the error of Vartia's algorithm vanishes linearly. They also demonstrate numerically that their own algorithm converges quadratically, much faster than Vartia's does. However, B&S's code of Vartia's algorithm contains errors.⁴ Largely based on this erroneous simulation, B&S suggest that their algorithm dominates Vartia's in convergence performance.

We find B&S's reasoning on the convergence rate of Vartia's algorithm to be flawed. B&S argue that Vartia's extended trapezoidal calculation in the innermost loops would reduce the vanishing speed of errors (i.e. the convergence rate with respect to n). However, these innermost loops are iterations for the CV to converge within each step, and the errors of the iterations can be controlled by the error tolerance of iterations. More importantly, these errors are not related to n ; that is, they do not affect the algorithm's convergence rate with respect to n .⁵

The approximated CV within each step of Vartia's algorithm is very close to the trapezoidal approximation of the integral of the Hicksian demand. Within each step, the error of the trapezoidal approximation is to the third order; therefore, the total error should be accumulated to the second order, not the first order or the third order. B&S's algorithm uses a second-order Taylor approximation, which also has third-order errors in each step, so the total error should be accumulated to the second order in their algorithm as well. Thus, the two algorithms have the same quadratic convergence rate with respect to n .

Here we present a theorem that Vartia's and B&S's algorithms both converge quadratically. To prove this theorem, we first provide the following lemmas constructing the bounds of the errors of the two algorithms.

⁴ For B&S's GAUSS code, see Table AII in B&S. In their code, the index j should have a starting value of 1 instead of 2. Moreover, the base demand update "xc = xt" (the eleventh lowest line) should be executed only if the within-step interaction has converged; that is, this line should be placed after the line "check for convergence".

⁵ The errors of the iterations affect the computing time of Vartia's algorithm given n . For discussions of the computing time of the two algorithms, see Footnote 11.

Lemma 1 Suppose the Marshallian demand $q(p, C)$ is well-behaved and thrice continuously differentiable. For any finite price change from p^0 to p^1 , divide the price change into n steps. The error of Vartia’s algorithm over k steps ($k < n$) is bounded by a term proportional to $1/n^3$. Mathematically,

$$|\varepsilon_k^V| \leq \frac{N_1}{12} \left(\frac{|\Delta p|}{n}\right)^2 \left[\left(\left(\frac{1 + \frac{K_1|\Delta p|}{2n}}{1 - \frac{K_1|\Delta p|}{2n}} \right)^k - 1 \right) \frac{1}{K_1} \right] \sim \frac{1}{n^3}, \tag{7}$$

where $\Delta p = p^1 - p^0$, K_1 and N_1 are constants.

Lemma 2 Suppose the Marshallian demand $q(p, C)$ is well-behaved and thrice continuously differentiable. For any finite price change from p^0 to p^1 , divide the price change into n steps. The error of B&S’s algorithm over k steps ($k < n$) is bounded by a term proportional to $1/n^3$. Mathematically,

$$|\varepsilon_k^{B\&S}| \leq \frac{N_1}{6} \left(\frac{|\Delta p|}{n}\right)^2 \left[\frac{\left(1 + K_1 \left(\frac{|\Delta p|}{n}\right) + \frac{1}{2}K_2 \left(\frac{|\Delta p|}{n}\right)^2\right)^k - 1}{K_1 + \frac{1}{2}K_2 \left(\frac{|\Delta p|}{n}\right)} \right] \sim \frac{1}{n^3}, \tag{8}$$

where $\Delta p = p^1 - p^0$, K_1 , K_2 , and N_1 are constants.

We prove Lemmas 1 and 2 in Appendices A.1 and A.2.

We proceed with the main theorem presenting the exact convergence rates and relative size of the errors.

Theorem 1 Suppose the Marshallian demand $q(p, C)$ is well-behaved and thrice continuously differentiable. For any finite price change from p^0 to p^1 , the calculated compensated incomes from both algorithms converge to the true compensated income quadratically. Mathematically,

$$\varepsilon_n^V = \frac{1}{12} \left(\frac{dq(p^1)}{dp} - \frac{dq(p^0)}{dp} \right) \left(\frac{\Delta p}{n} \right)^2 + o\left(\left(\frac{\Delta p}{n} \right)^2 \right) \sim \frac{1}{n^2}, \tag{9}$$

and

$$\varepsilon_n^{B\&S} = -\frac{1}{6} \left(\frac{dq(p^1)}{dp} - \frac{dq(p^0)}{dp} \right) \left(\frac{\Delta p}{n} \right)^2 + o\left(\left(\frac{\Delta p}{n} \right)^2 \right) \sim \frac{1}{n^2}, \tag{10}$$

where $\frac{dq(p^0)}{dp} = \frac{dq(p^0, C(p^0))}{dp}$ and $\frac{dq(p^1)}{dp} = \frac{dq(p^1, C(p^1))}{dp}$. Therefore, if the price change within each partition step is small, i.e., if $\frac{\Delta p}{n}$ is small, then the error of Vartia’s algorithm is approximately half that of B&S’s algorithm, and the errors have opposite signs. Mathematically,

$$\varepsilon_n^V = -\frac{1}{2}\varepsilon_n^{B\&S} + o\left(\left(\frac{\Delta p}{n}\right)^2\right). \quad (11)$$

We prove Theorem 1 in Appendix A.3.

4 Simulations

4.1 Three Demand Systems

We use three calibrated demand systems, with one-good, two-good, and three-good price changes, to compare the approximated CVs by the two algorithms. To make the results comparable with those in B&S, we use the same demand systems that B&S use.

The one-good demand system is the linear Marshallian demand system discussed by Hausman (1981). The demand function and the indirect utility function are in the following forms:

$$q(p, C) = \alpha p + \delta C + \gamma, \quad (12)$$

and

$$v(p, C) = e^{-\delta p} \left(C + \frac{1}{\delta} \left(\alpha p + \frac{\alpha}{\delta} + \gamma \right) \right). \quad (13)$$

Hausman (1981) sets the income for the mean person at \$720 per month, and the initial price at \$0.75. The demand parameters are set at $\alpha = -14.22$, $\delta = 0.082$, and $\gamma = 4.95$. We analyze the welfare effect of a rise in price.

The second demand system is the two-good indirect addilog demand (IAD) system discussed by McKenzie and Pearce (1976).⁶ The demand functions and the indirect utility function are in the following forms:

$$q_i(p, C) = \frac{\beta_i p_j}{p_i} \frac{C}{\beta_1 p_2 + \beta_2 p_1}, \quad i \neq j, \quad i, j = 1, 2, \quad (14)$$

and

$$v(p, C) = C \left(\frac{\beta_1}{p_1} + \frac{\beta_2}{p_2} \right). \quad (15)$$

McKenzie and Pearce (1976) set the income at \$220 and the initial prices at (1.0, 2.0). The demand parameters are set at $\beta_1 = \beta_2 = 1$.

The third demand system is the three-good IAD system discussed by B&S. The demand functions and the indirect utility function are given by the equations

⁶ For examples and discussion of the IAD model, see Houthakker (1960), Yoshihara (1969), and Parks (1969).

Table 1 CV estimates from different sources

Algorithm	Source of results	$n = 4$	$n = 5$	$n = 8$	$n = 20$
Vartia	B&S's simulation		0.02		0.01
	Our replication of B&S's code	0.0241	0.0207	0.0142	0.0060
	Vartia's calculation	-0.0083		-0.0024	
	Our simulation	-0.0083	-0.0054	-0.0024	-0.0007
B&S	B&S's simulation		0.01		0.00
	Our simulation	0.0150	0.0095	0.0035	0.0002

The two-good IAD system: $q_i(p, C) = \frac{\beta_i p_j}{p_i} \frac{C}{\beta_1 p_2 + \beta_2 p_1}$, $i \neq j$, $i, j = 1, 2$ $\beta_1 = \beta_2 = 1$, $p^0 = (1.0, 2.0)$, $p^1 = (1.1, 1.6923)$, $C^0 = 220$

The true CV for the given price change is -0.0004

$$q_i(p, C) = \frac{\alpha_i \beta_i y^{\beta_i} p_i^{-\beta_i - 1}}{\sum_{i=1}^3 \alpha_i \beta_i y^{\beta_i - 1} p_i^{-\beta_i}}, \quad i = 1, 2, 3, \tag{16}$$

and

$$v(p, C) = \sum_{i=1}^3 \alpha_i \left(\frac{y}{p_i}\right)^{\beta_i}. \tag{17}$$

We use the parameters, initial prices, and income provided by B&S, as $\alpha = (1.5, 1.5, 1.2)$, $\beta = (1.0, 0.8, 1.2)$, $p^0 = (5.0, 5.0, 4.0)$, and $C^0 = 1000$.

4.2 Examination of Programming Routines

Vartia and B&S both use the two-good IAD system to illustrate their algorithms by moving along the indifference surface from the initial prices (1.0, 2.0) to the final prices (1.1, 1.6923). Vartia calculates the CV using the number of partition steps $n = 4$ and $n = 8$; B&S calculate the CV using $n = 5$, $n = 20$, and $n = 100$. This demand system and the parameters specification allow us to examine the simulation programs used in B&S and our paper. We compare the CV obtained in Vartia's and B&S's papers with the results of our replication of their codes and our own simulation.⁷

Table 1 shows the simulation results from the different sources. When we use B&S's code for simulation of Vartia's algorithm, our results are consistent with that shown in B&S, but different from Vartia's calculation.⁸ Our simulation of Vartia's algorithm matches the result of Vartia's calculation. We conclude that B&S

⁷ All replications and simulations are programmed and run in MATLAB. The error tolerance for iteration in Vartia's algorithm is set at 10^{-8} . Our codes are available upon request.

⁸ For the results shown in B&S, see Table II in B&S. For the results in Vartia, see Tables II and III in Vartia.

make mistakes in their simulation and that our program of Vartia's algorithm is valid.

Our simulation of B&S's algorithm shows consistent results with those in B&S, verifying our program of B&S's algorithm.⁹

Given the specific demand system and price change, the true CV is -0.0004 , very close to zero; Vartia's algorithm reports the right sign, while B&S's algorithm does not. This confirms that errors in numerical algorithms may lead to erroneous conclusions about whether a consumer will be better off or worse off after a change in state.

Table 1 also shows that the approximated CV by Vartia's algorithm is closer to the true CV at the given numbers of partition steps. In the following subsections we will show that in the three demand systems this is true when the number of partition steps is large, and we will quantify and compare the convergence rates and error sizes of the two algorithms.

4.3 Convergence Rates

For each of the three demand systems, we calculate the true value of CV given a specific price change, and we record errors of the two algorithms at different numbers of partition steps for the specific price change. To numerically verify the quadratic convergence rates of the two algorithms, we consider the relationship between the step size ($|\Delta p|/n$) and the absolute error ($|\varepsilon|$) with the form considered by B&S:

$$|\varepsilon| = K \left(\frac{|\Delta p|}{n} \right)^b, \quad (18)$$

and hence,

$$\ln(|\varepsilon|) = a + b \ln \left(\frac{|\Delta p|}{n} \right), \quad (19)$$

where $|\Delta p|$ is the Euclidean norm of the price change. We estimate a and b using the OLS estimator. The estimate of b gives the order of the error term (i.e., the rate of convergence); the estimate of a (hence K) gives the constant proportionality of the absolute error. The algorithm with a larger b and lower a (therefore, a smaller K) is more accurate. We run regressions for the number of partition steps n from 2 to 100 for the two algorithms in each of the three demand systems and compare the estimates of a and b .¹⁰ The results are shown in Table 2.

The estimates of b for both algorithms in all demand systems are very close to 2 (with the difference within 0.01). This result indicates that both algorithms converge quadratically with respect to n , confirming Theorem 1. Moreover, since Vartia's

⁹ For the results shown in B&S, see Table II in B&S.

¹⁰ In the two-good and three-good cases, B&S use equal price changes for all goods. To increase the generalizability of our results, we use different price changes for different goods.

Table 2 Estimates of convergence performance of the two algorithms

Algorithm	Parameters	One good $\Delta p = 0.5$	Two goods $\Delta p = (0.25, 0.75)$	Three goods $\Delta p = (0.25, 0.75)$
Vartia	\hat{a}	-3.3509 (0.0000)	-1.9963 (0.0006)	-1.3186 (0.0003)
	\hat{b}	2.0000 (0.0000)	1.9988 (0.0002)	1.9993 (0.0001)
	R^2	1.0000	1.0000	1.0000
B&S	\hat{a}	-2.6681 (0.0006)	-1.2519 (0.0024)	-0.6187 (0.0002)
	\hat{b}	1.9979 (0.0001)	2.0101 (0.0006)	2.0012 (0.0001)
	R^2	1.0000	1.0000	1.0000

The model: $\ln(|\varepsilon|) = a + b \ln\left(\frac{|\Delta p|}{n}\right)$

All models are estimated with the OLS estimator. The number of observations is 99 in all models. Standard errors of estimates are in parenthesis

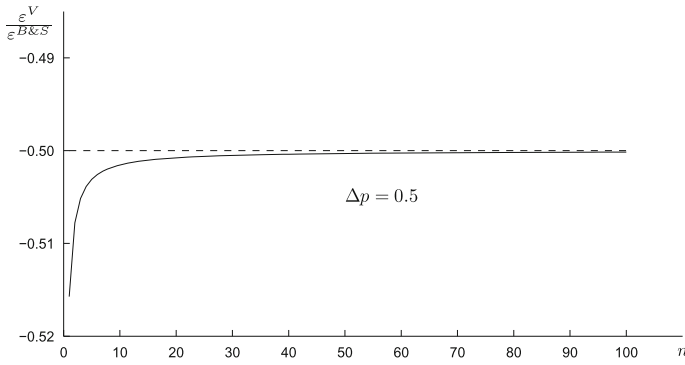
algorithm has considerably lower \hat{a} , it is likely to dominate B&S’s algorithm in convergence performance.

4.4 Relative Size of the Errors

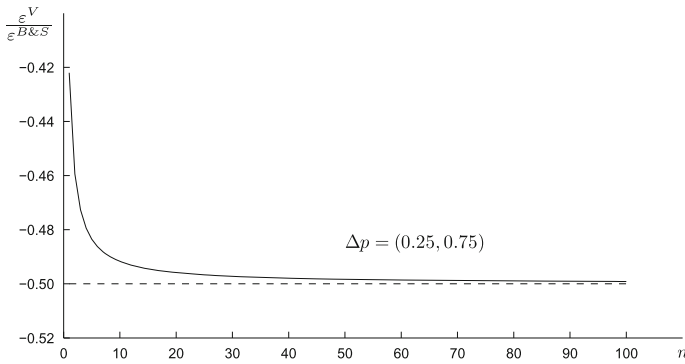
To investigate the relative size of the two algorithms’ errors, and to confirm the dominance of Vartia’s algorithm, we plot $\varepsilon^V / \varepsilon^{B\&S}$, the error ratio of the two algorithms, with respect to n , the number of partition steps in Fig. 2.

The error ratios in the three demand systems all converge to -0.5 as n increases. This result confirms Theorem 1, that when $\Delta p/n$ is small, the error of Vartia’s algorithm is approximately half that of B&S’s algorithm and the errors have opposite signs.¹¹

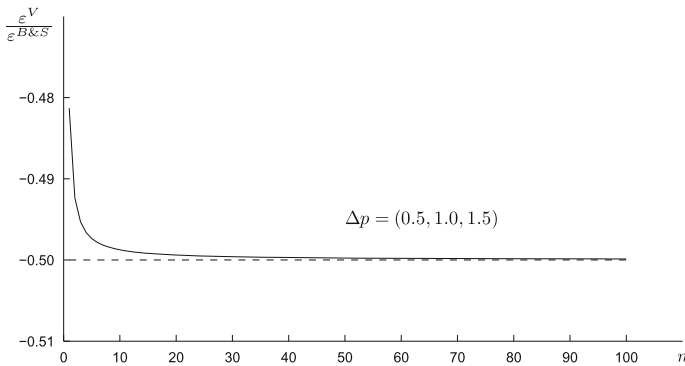
¹¹ A concern with numerical methods is the computing time. We test the computing time with simulations of the IAD system $q_i = C / \left(p_i^2 \cdot \sum_{i=1}^d p_i^{-1} \right)$, where i is the index of the good and d is the number of goods. We conduct simulations for d from 1 to 10. For each d , we generate a d -dimensional price change Δp with Δp_i generated (independently) as a random number in $(-1, 1)$. Our simulations suggest that Vartia’s algorithm gets faster compared to B&S’s algorithm as the numbers of goods (d) and partition steps (n) increase. For example, at $d = 6$, Vartia’s algorithm is faster when $n \geq 100$; at $n = 50$, Vartia’s algorithm is faster when $d \geq 7$; Vartia’s algorithm is always faster than B&S’s algorithm for $d \geq 8$. The results are not surprising. As the number of goods rises, the Slutsky matrix used in B&S’s algorithm becomes increasingly complicated compared to the Marshallian demand function, which is the only functional form used in Vartia’s algorithm. As the number of partition steps increases, the step size ($|\Delta p|/n$) becomes smaller, so each within-step iteration in Vartia’s algorithm takes shorter time to converge, while B&S’s algorithm does not enjoy this advantage. Therefore, as the numbers of goods and partition steps increase, Vartia’s algorithm is more likely to dominate B&S’s algorithm in computing time. The results are available upon request.



(a) One-good linear demand system



(b) Two-good IAD system



(c) Three-good IAD system

Fig. 2 The error ratio with respect to the number of partition steps

5 Conclusion

Vartia's and B&S's algorithms are popular approaches for approximating compensated income numerically. B&S propose that their algorithm is superior in accuracy

and convergence performance. Motivated by a one-step example with graphical analysis, we prove the theorem that in well-behaved demand systems, both algorithms converge quadratically, and when the price change within each partition step is small, the error of Vartia’s algorithm is approximately half that of B&S’s algorithm. Simulations with one-good and multi-good cases confirm our theorem. Therefore, we correct the claims in both Vartia’s and B&S’s papers and conclude that Vartia’s algorithm is more accurate than B&S’s algorithm.

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Appendix A: Proofs of the Lemmas and the Theorem

We denote the compensated income at price p_k as $C(p_k)$; the compensated demand at price p_k is $q(p_k, C(p_k)) \equiv q(p_k)$, and, by Shephard’s Lemma, $\frac{dC}{dp} = q(p)$ holds. We denote the calculated compensated income at price p_k as C_k ; the calculated compensated demand at price p_k is therefore $q(p_k, C_k) \equiv q_k$. For any finite price change from p^0 to p^1 , the Marshallian demand $q(p, C)$ is thrice continuously differentiable, so the Lipschitz condition is satisfied:

$$|q(p, C_1) - q(p, C_2)| \leq K_1|C_1 - C_2|, \tag{20}$$

and the second-order derivative of $q(p)$ is bounded:

$$\left| \frac{d^2q(p)}{dp^2} \right| \leq N_1. \tag{21}$$

We denote the error term over k steps as ε_k , i.e., $\varepsilon_k = C_k - C(p_k)$. Following the idea presented in Collatz (1960), we consider the error increase at each step.

A.1 Lemma 1: The Bound of the Error of Vartia’s Algorithm

Vartia’s algorithm is based on Eq. (3):

$$C_k - C_{k-1} = \frac{1}{2}(q_k + q_{k-1})\Delta p_k, \tag{22}$$

where $k = 1, \dots, n$, $q_k = q(p_k, C_k)$, $q_{k-1} = q(p_{k-1}, C_{k-1})$, $\Delta p_k = p_k - p_{k-1} = \Delta p/n$, and the starting values are $(p_0, q_0, C_0) = (p^0, q(p^0, C^0), C^0)$. Therefore, $\varepsilon_0 = C_0 - C(p_0) = 0$, where $C(p_0)$ is the given initial income. Vartia proves the convergence of the iteration within each step when the step size is small, so Eq. (3) is well defined.¹²

¹² For Vartia’s proof, see Appendix 2 in Vartia.

The total error over k steps can be calculated as $\varepsilon_k = C_k - C(p_k)$. Denote the increase in the error from step $k - 1$ to step k as $\Delta\varepsilon_k = \varepsilon_k - \varepsilon_{k-1}$. Therefore, we have

$$\begin{aligned} \Delta\varepsilon_k &= \varepsilon_k - \varepsilon_{k-1} = (C_k - C_{k-1}) - (C(p_k) - C(p_{k-1})) \\ &= \frac{1}{2} (q_k + q_{k-1}) \Delta p_k - \int_{p_{k-1}}^{p_k} q(p) dp \\ &= \frac{1}{2} [(q_k - q(p_k)) + (q_{k-1} - q(p_{k-1}))] \Delta p_k + \frac{1}{2} (q(p_k) + q(p_{k-1})) \Delta p_k \\ &\quad - \int_{p_{k-1}}^{p_k} q(p) dp. \end{aligned} \tag{23}$$

By Eq. (20), we get

$$|q_k - q(p_k)| \leq K_1 |C_k - C(p_k)| = K_1 |\varepsilon_k|, \tag{24}$$

and

$$|q_{k-1} - q(p_{k-1})| \leq K_1 |C_{k-1} - C(p_{k-1})| = K_1 |\varepsilon_{k-1}|. \tag{25}$$

By the well-known Trapezoidal rule, we know

$$\left| \frac{1}{2} (q(p_k) + q(p_{k-1})) \Delta p_k - \int_{p_{k-1}}^{p_k} q(p) dp \right| \leq N_1 \frac{|\Delta p_k|^3}{12} = \frac{N_1}{12} \left(\frac{|\Delta p|}{n} \right)^3, \tag{26}$$

where N_1 is the bound of the second-order derivative of $q(p, C(p))$.

Plug these bounds into Eq. (23), we get

$$|\Delta\varepsilon_k| = |\varepsilon_k - \varepsilon_{k-1}| \leq \frac{K_1 |\Delta p|}{2n} (|\varepsilon_k| + |\varepsilon_{k-1}|) + \frac{N_1}{12} \left(\frac{|\Delta p|}{n} \right)^3. \tag{27}$$

We know $|\varepsilon_k| - |\varepsilon_{k-1}| \leq |\varepsilon_k - \varepsilon_{k-1}|$; therefore, for a large n the equation can be rearranged as follows:

$$\left(1 - \frac{K_1 |\Delta p|}{2n} \right) |\varepsilon_k| \leq \left(1 + \frac{K_1 |\Delta p|}{2n} \right) |\varepsilon_{k-1}| + \frac{N_1}{12} \left(\frac{|\Delta p|}{n} \right)^3, \tag{28}$$

and hence,

$$|\varepsilon_k| \leq \frac{1 + \frac{K_1 |\Delta p|}{2n}}{1 - \frac{K_1 |\Delta p|}{2n}} |\varepsilon_{k-1}| + \frac{\frac{N_1}{12} \left(\frac{|\Delta p|}{n} \right)^3}{1 - \frac{K_1 |\Delta p|}{2n}}. \tag{29}$$

Here, $\varepsilon_0 = 0$. With some basic algebra, we estimate the error of Vartia’s algorithm over k steps as follows:

$$\begin{aligned}
 |\varepsilon_k| &\leq \frac{N_1}{12} \left(\frac{|\Delta p|}{n}\right)^2 \left[\left(\left(\frac{1 + \frac{K_1|\Delta p|}{2n}}{1 - \frac{K_1|\Delta p|}{2n}} \right)^k - 1 \right) \frac{1}{K_1} \right] \\
 &= \frac{N_1}{12} \left(\frac{|\Delta p|}{n}\right)^2 \left[\frac{1 + kK_1 \frac{|\Delta p|}{n} + o\left(\frac{|\Delta p|}{n}\right) - 1}{K_1} \right] \\
 &= k \frac{N_1}{12} \left(\frac{|\Delta p|}{n}\right)^3 + o\left(\left(\frac{|\Delta p|}{n}\right)^3\right) \sim \frac{1}{n^3},
 \end{aligned}
 \tag{30}$$

where $k < n$. Therefore, the error over k steps is bounded by a term proportional to $1/n^3$. □

A.2 Lemma 2: The Bound of the Error of B&S’s Algorithm

B&S’s algorithm is based on Eq. (5)

$$C_k - C_{k-1} = q_{k-1} \Delta p_k + \frac{1}{2} \left(\frac{\partial q_{k-1}}{\partial p} + \frac{\partial q_{k-1}}{\partial C} q_{k-1} \right) (\Delta p_k)^2,
 \tag{31}$$

where $k = 1, \dots, n, q_{k-1} = q(p_{k-1}, C_{k-1}), \Delta p_k = p_k - p_{k-1} = \Delta p/n$, and the starting values are $(p_0, q_0, C_0) = (p^0, q(p^0, C^0), C^0)$.

Define a new function $g(p, C) = \frac{\partial q(p,C)}{\partial p} + \frac{\partial q(p,C)}{\partial C} q(p, C)$. Therefore, we have

$$g_k \equiv g(p_k, C_k) = \frac{\partial q_k}{\partial p} + \frac{\partial q_k}{\partial C} q_k,
 \tag{32}$$

and

$$\begin{aligned}
 g(p_k) \equiv g(p_k, C(p_k)) &= \frac{\partial q(p_k)}{\partial p} + \frac{\partial q(p_k)}{\partial C} q(p_k) = \frac{\partial q(p_k)}{\partial p} + \frac{\partial q(p_k)}{\partial C} \frac{\partial C(p_k)}{\partial p} \\
 &= \frac{dq(p_k)}{dp}.
 \end{aligned}
 \tag{33}$$

Note that $g(p_k) = \frac{dq(p_k)}{dp}$, while $g(p_k, C_k) \neq \frac{dq_k}{dp}$. As C_k is our calculated value for the compensated income at price p_k , the term dC_k/dp is not defined and we do not have the relation $dC_k/dp = q(p_k, C_k)$.

By the thrice continuous differentiability of the function $q(p, C)$, we know that $g(p, C)$ is twice continuously differentiable and has bounded partial derivative with respect to C for any finite change from C_0 to C_1 . Therefore, it satisfies the Lipschitz condition:

$$|g(p, C_1) - g(p, C_2)| \leq K_2|C_1 - C_2|. \tag{34}$$

Using the same notations as in Lemma 1’s proof, together with Eqs. (32) and (33), we can formulate the increase in the error as follows:

$$\Delta\varepsilon_k = \varepsilon_k - \varepsilon_{k-1} = (C_k - C_{k-1}) - (C(p_k) - C(p_{k-1})). \tag{35}$$

Use Eqs. (5) and (32) on the first term, and expand the second term at p_{k-1} :

$$\begin{aligned} \Delta\varepsilon_k &= \left[q_{k-1}\Delta p_k + \frac{1}{2}g_{k-1}(\Delta p_k)^2 \right] \\ &\quad - \left[q(p_{k-1})(\Delta p_k) + \frac{1}{2}\frac{dq(p_{k-1})}{dp}(\Delta p_k)^2 + \frac{1}{6}\frac{d^2q(p^*)}{dp^2}(\Delta p_k)^3 \right] \\ &= (q_{k-1} - q(p_{k-1}))\Delta p_k + \frac{1}{2}(g_{k-1} - g(p_{k-1}))(\Delta p_k)^2 - \frac{1}{6}\frac{d^2q(p^*)}{dp^2}(\Delta p_k)^3, \end{aligned} \tag{36}$$

where $p^* \in [p_{k-1}, p_k]$.

Keep in mind that $g(p_k) = \frac{dq(p_k)}{dp}$. Therefore, by Eqs. (20), (34), and (21) we can get

$$\begin{aligned} |q_{k-1} - q(p_{k-1})| &\leq K_1|C_{k-1} - C(p_{k-1})| = K_1|\varepsilon_{k-1}|, \\ |g_{k-1} - g(p_{k-1})| &\leq K_2|C_{k-1} - C(p_{k-1})| = K_2|\varepsilon_{k-1}|, \end{aligned} \tag{37}$$

and

$$\left| \frac{1}{6}\frac{d^2q(p^*)}{dp^2}(\Delta p_k)^3 \right| \leq \frac{N_1}{6}|(\Delta p_k)^3| = \frac{N_1}{6}\left(\frac{|\Delta p|}{n}\right)^3. \tag{38}$$

Therefore, using methods similar to the proof of Lemma 1, we get

$$|\varepsilon_k| - |\varepsilon_{k-1}| \leq |\varepsilon_k - \varepsilon_{k-1}| \leq K_1|\varepsilon_{k-1}|\left(\frac{|\Delta p|}{n}\right) + \frac{1}{2}K_2|\varepsilon_{k-1}|\left(\frac{|\Delta p|}{n}\right)^2 + \frac{N_1}{6}\left(\frac{|\Delta p|}{n}\right)^3, \tag{39}$$

and hence,

$$|\varepsilon_k| \leq \left[1 + K_1\left(\frac{|\Delta p|}{n}\right) + \frac{1}{2}K_2\left(\frac{|\Delta p|}{n}\right)^2 \right] |\varepsilon_{k-1}| + \frac{N_1}{6}\left(\frac{|\Delta p|}{n}\right)^3. \tag{40}$$

We can solve for the estimate of the error over k steps as

$$\begin{aligned}
 |\varepsilon_k| &\leq \frac{N_1}{6} \left(\frac{|\Delta p|}{n}\right)^2 \left[\frac{\left(1 + K_1 \left(\frac{|\Delta p|}{n}\right) + \frac{1}{2} K_2 \left(\frac{|\Delta p|}{n}\right)^2\right)^k - 1}{K_1 + \frac{1}{2} K_2 \left(\frac{|\Delta p|}{n}\right)} \right] \\
 &= \frac{N_1}{6} \left(\frac{|\Delta p|}{n}\right)^2 \left[\frac{k K_1 \cdot K_1 |\Delta p|}{K_1^2 n} + o\left(\frac{|\Delta p|}{n}\right) \right] \\
 &= k \frac{N_1}{6} \left(\frac{|\Delta p|}{n}\right)^3 + o\left(\left(\frac{|\Delta p|}{n}\right)^3\right) \sim \frac{1}{n^3},
 \end{aligned}
 \tag{41}$$

where $k < n$. Therefore, the error over k steps is bounded by a term proportional to $1/n^3$. □

A.3 Theorem 1: The Convergence Rates and Relative Size of the Errors

Here, we prove that both algorithms converge quadratically and that the error of Vartia’s algorithm is approximately half that of B&S’s algorithm when the price change within each partition step is small.

First consider Vartia’s algorithm. In Eq. (23),

$$\begin{aligned}
 \Delta\varepsilon_k^V &= \frac{1}{2} [(q_k - q(p_k)) + (q_{k-1} - q(p_{k-1}))] \Delta p_k \\
 &\quad + \frac{1}{2} (q(p_k) + q(p_{k-1})) \Delta p_k - \int_{p_{k-1}}^{p_k} q(p) dp,
 \end{aligned}
 \tag{42}$$

where $q_k = q(p_k, C_k)$, $q(p_k) = q(p_k, C(p_k))$, $q_{k-1} = q(p_{k-1}, C_{k-1})$, $q(p_{k-1}) = q(p_{k-1}, C(p_{k-1}))$, $q(p) = q(p, C(p))$, and $\Delta p_k = p_k - p_{k-1}$. Expand at $C(p_k)$ and $C(p_{k-1})$ respectively for terms in the first $[\cdot]$, expand at p_{k-1} for the remaining terms, and combine the like terms:

$$\begin{aligned}
 \Delta\varepsilon_k^V &= \frac{1}{2} \left[\frac{\partial q(p_k)}{\partial C} \varepsilon_k + o(\varepsilon_k) + \frac{\partial q(p_{k-1})}{\partial C} \varepsilon_{k-1} + o(\varepsilon_{k-1}) \right] \Delta p_k \\
 &\quad + \frac{1}{2} \left[2q(p_{k-1}) + \frac{dq(p_{k-1})}{dp} \Delta p_k + \frac{1}{2} \frac{d^2q(p_{k-1})}{dp^2} (\Delta p_k)^2 + o\left((\Delta p_k)^2\right) \right] \Delta p_k \\
 &\quad - \left[q(p_{k-1}) \frac{\Delta p}{n} + \frac{1}{2} \frac{dq(p_{k-1})}{dp} (\Delta p_k)^2 + \frac{1}{6} \frac{d^2q(p_{k-1})}{dp^2} (\Delta p_k)^3 + o\left((\Delta p_k)^3\right) \right] \\
 &= \frac{1}{2} \left[\frac{\partial q(p_k)}{\partial C} \varepsilon_k + \frac{\partial q(p_{k-1})}{\partial C} \varepsilon_{k-1} \right] \Delta p_k + \frac{1}{12} \frac{d^2q(p_{k-1})}{dp^2} (\Delta p_k)^3 + o\left((\Delta p_k)^3\right).
 \end{aligned}
 \tag{43}$$

Using the result of Lemma 1 that ε_k is bounded by a term proportional to $1/n^3$, we get

$$\Delta \varepsilon_k^V = \frac{1}{12} \frac{d^2 q(p_{k-1})}{dp^2} \left(\frac{\Delta p}{n}\right)^3 + o\left(\left(\frac{\Delta p}{n}\right)^3\right). \tag{44}$$

Since $\varepsilon_0^V = 0$, we have

$$\begin{aligned} \varepsilon_n^V &= \sum_{k=1}^n (\Delta \varepsilon_k^V) \\ &= \sum_{k=1}^n \left[\frac{1}{12} \frac{d^2 q(p_{k-1})}{dp^2} \left(\frac{\Delta p}{n}\right)^3 + o\left(\left(\frac{\Delta p}{n}\right)^3\right) \right] \\ &= \frac{1}{12} \left(\frac{\Delta p}{n}\right)^2 \sum_{k=1}^n \left[\frac{d^2 q(p_{k-1})}{dp^2} \frac{\Delta p}{n} \right] + \sum_{k=1}^n o\left(\left(\frac{\Delta p}{n}\right)^3\right) \\ &= \frac{1}{12} \left(\frac{\Delta p}{n}\right)^2 \sum_{k=1}^n \left[\frac{d^2 q(p_{k-1})}{dp^2} \frac{\Delta p}{n} \right] + o\left(\left(\frac{\Delta p}{n}\right)^2\right) \\ &= \frac{1}{12} \left(\frac{\Delta p}{n}\right)^2 \int_{p^0}^{p^1} \frac{d^2 q(p)}{dp^2} dp + o\left(\left(\frac{\Delta p}{n}\right)^2\right) \\ &= \frac{1}{12} \left(\frac{\Delta p}{n}\right)^2 \left(\frac{dq(p^1)}{dp} - \frac{dq(p^0)}{dp} \right) + o\left(\left(\frac{\Delta p}{n}\right)^2\right) \\ &\sim \frac{1}{n^2}. \end{aligned} \tag{45}$$

Therefore, the quadratic convergence rate of Vartia’s algorithm is proved.

Second consider B&S’s algorithm. In Eq. (36),

$$\begin{aligned} \Delta \varepsilon_k^{B\&S} &= (q_{k-1} - q(p_{k-1})) \Delta p_k + \frac{1}{2} (g_{k-1} - g(p_{k-1})) (\Delta p_k)^2 \\ &\quad - \frac{1}{6} \frac{d^2 q(p_{k-1})}{dp^2} (\Delta p_k)^3 + o\left((\Delta p_k)^3\right), \end{aligned} \tag{46}$$

where $q_{k-1} = q(p_{k-1}, C_{k-1})$, $q(p_{k-1}) = q(p_{k-1}, C(p_{k-1}))$, $g_{k-1} = g(p_{k-1}, C_{k-1})$, $g(p_{k-1}) = g(p_{k-1}, C(p_{k-1}))$, and $\Delta p_k = p_k - p_{k-1}$. Expanding at $C(p_{k-1})$, we get

$$\begin{aligned} \Delta \varepsilon_k^{B\&S} &= \left[\frac{\partial q(p_{k-1})}{\partial C} \varepsilon_{k-1} + o(\varepsilon_{k-1}) \right] \Delta p_k + \frac{1}{2} \left[\frac{\partial g(p_{k-1})}{\partial C} \varepsilon_{k-1} + o(\varepsilon_{k-1}) \right] (\Delta p_k)^2 \\ &\quad - \frac{1}{6} \frac{d^2 q(p_{k-1})}{dp^2} (\Delta p_k)^3 + o\left((\Delta p_k)^3\right). \end{aligned} \tag{47}$$

Similarly, using the result of Lemma 2 that ε_k is bounded by a term proportional to $1/n^3$, we get

$$\Delta \varepsilon_k^{B\&S} = -\frac{1}{6} \frac{d^2 q(p_{k-1})}{dp^2} \left(\frac{\Delta p}{n}\right)^3 + o\left(\left(\frac{\Delta p}{n}\right)^3\right). \quad (48)$$

Since $\varepsilon_0^{B\&S} = 0$, similarly we have

$$\begin{aligned} \varepsilon_n^{B\&S} &= \sum_{k=1}^n \left(\Delta \varepsilon_k^{B\&S}\right) \\ &= \sum_{k=1}^n \left[-\frac{1}{6} \frac{d^2 q(p_{k-1})}{dp^2} \left(\frac{\Delta p}{n}\right)^3 + o\left(\left(\frac{\Delta p}{n}\right)^3\right) \right] \\ &= -\frac{1}{6} \left(\frac{\Delta p}{n}\right)^2 \int_{p^0}^{p^1} \frac{d^2 q(p)}{dp^2} dp + o\left(\left(\frac{\Delta p}{n}\right)^2\right) \\ &= -\frac{1}{6} \left(\frac{\Delta p}{n}\right)^2 \left(\frac{dq(p^1)}{dp} - \frac{dq(p^0)}{dp}\right) + o\left(\left(\frac{\Delta p}{n}\right)^2\right) \\ &\sim \frac{1}{n^2}. \end{aligned} \quad (49)$$

Therefore, the quadratic convergence rate of B&S's algorithm is proved.

By the two last equalities in Eqs. (45) and (49), $\varepsilon_n^V = -\frac{1}{2}\varepsilon_n^{B\&S} + o\left(\left(\frac{\Delta p}{n}\right)^2\right)$.

Therefore, if $\frac{\Delta p}{n}$ is small, the error of Vartia's algorithm is approximately half that of B&S's algorithm, and they have opposite signs. \square

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